Math 4200
Monday September 28 2.1-2.2 recap, and technical discussion of "connected" vs. "path connected".

Announcements: Modified hw5 and early view of hw6!

Warm-up exercise

hw5 Due Friday October 2 at 11:59 p.m.

Do the following problems using the Theorems from section 2.1-2.2. These include the *FTC Theorem* 2.1.7; *Cauchy's Theorem* 2.2.1 and 2.2.3; the *Deformation Theorem* 2.2.2 which we also call the *Replacement Theorem* in class; the *Antiderivative Theorem* 2.2.3 which we make rigorous in section 2.3.

2.2 : 5, 11.
2.3 7, 10.

hw6 Due Wednesday October 7 at 11:59 p.m. (Section 2.3 is potentially on the Friday October 9 midterm.)

Do the following problems using the Theorems and definitions from section 2.3. These include the definitions of *homotopies with fixed endpoints* 2.3.6; and *homotopies of closed curves* 2.3.7; the precise (homotopy) definition of *simply connected* 2.3.8; the homotopy versions of the *Deformation Theorem* 2.3.12 and *Cauchy's Theorem* 2.3.14; the rigorous *Antiderivative Theorem* 2.2.3 which is stated in section 2.2 but made rigrous in section 2.3.

2.3 1, 3, 5, 6, 7abc (This week show that each γ is homotopic to a point (contractible) in the domain of analyticity for f, so each integral is zero.), 9. In 9b write down a homotopy from the given curve to the standard parameterization of the unit circle, in $\mathbb{C}\backslash\{0\}$, to justify your work.

w6.1 *Positive distance lemma*: Prove that if $K \subseteq \mathbb{C}$ is *compact*, and if $K \subseteq \mathcal{O}$, where $\mathcal{O}$ is open, then there exists an $\varepsilon > 0$ such that for each $z \in K$, $D(z; \varepsilon) \subseteq \mathcal{O}$. This is equivalent to Distance Lemma 1.4.21 in the text. See if you can construct a proof without looking there first, but in any case write a proof in your own words. Recall that there are two definitions of *compact*, which are equivalent in $\mathbb{R}^n$:

(i) $K \subseteq \mathbb{R}^n$ is compact if and only if every *open cover* of K has a finite subcover.
or
(ii) $K \subseteq \mathbb{R}^n$ is compact if and only if every *sequence* in K has a subsequence which converges to a point in K.

(In $\mathbb{R}^n$ another characterization of *compact* is closed and bounded, but this characterization does not generalize to *metric spaces.*)

Review and Summary of Chapter 2 Theorems so far, for contour integrals. I'll use the text numbering and we'll briefly recall *why* each theorem is true.

Theorem 2.1.7 (*Fundamental Theorem of Calculus*)
Let $A \subseteq \mathbb{C}$ open, $f: A \rightarrow \mathbb{C}$ continuous, $\gamma: [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{C}$ a piecewise C^1 curve. If f has an analytic antiderivative in A, i.e. $F' = f$, then complex line integrals only depend on the endpoints of the curve γ, via the formula

$$\int_\gamma f(z)\, \mathrm{d}z := F(\gamma(b)) - F(\gamma(a))$$

Theorem 2.1.9 *(Path Independence Theorem)*
The following are equivalent, for $f: A \rightarrow \mathbb{C}$ continuous, where A is open and connected:

(i) $\exists\, F: A \rightarrow \mathbb{C}$ such that $F' = f$ on A

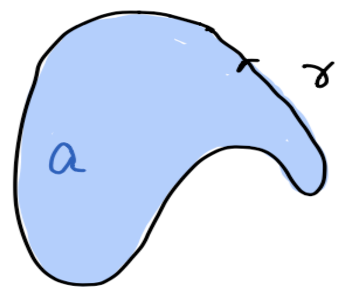
(ii) Contour integrals are *path independent*, i.e. for all choices of initial point P and terminal point Q in A,

$$\int_{\gamma_0} f(z)\, \mathrm{d}z = \int_{\gamma_1} f(z)\, \mathrm{d}z$$

whenenver γ_0, γ_1 are piecewise C^1 (continuous) paths that start at P and end at Q.

Theorem 2.2.1 *(Cauchy's Theorem)* Let γ be a simple closed piecewise C^1 contour, and let A be the bounded region inside of it. *If* $f(z)$ is C^1 and analytic in (a domain containing the closure of) A, then

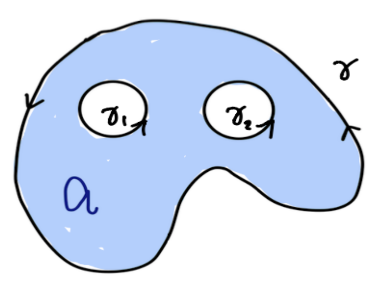
$$\int_{\gamma} f(z)\ \mathrm{d}z = 0.$$



Theorem 2.2.2 *(Replacement Theorem).* The text also calls this *a preliminary version of the deformation theorem,* which we discuss precisely in section 2.3.

Let $\gamma_1, \gamma_2, \ldots \gamma_n$ be non-overlapping simple closed curves such that γ is a simple closed curve with f analytic in the region between γ and $\gamma_1, \gamma_2, \ldots \gamma_n$ as indicated below. Orient all contours in the counterclockwise definition. Then

$$\int_{\gamma} f(z)\ \mathrm{d}z = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\ \mathrm{d}z$$



Example: (similar to some hw for this week) Let γ be the circle of radius 2 centered at the origin (and oriented counterclockwise as usual). Find

$$\int_{\gamma} \frac{z}{z^2 - 1} \, dz \, .$$

Combining *Cauchy's Theorem* and the *Path Independence Theorem* yields the result we were in the midst of proving at the very end of Friday's class:

Definition Let A be an open, connected domain. Then in section 2.2, A is called *simply connected* if it contains no holes. Another way to think about simply connected, which is closer to the precise definition in section 2.3, is that A is simply connected means that every closed contour in A can be continuously deformed into a constant (point) contour without ever leaving A.

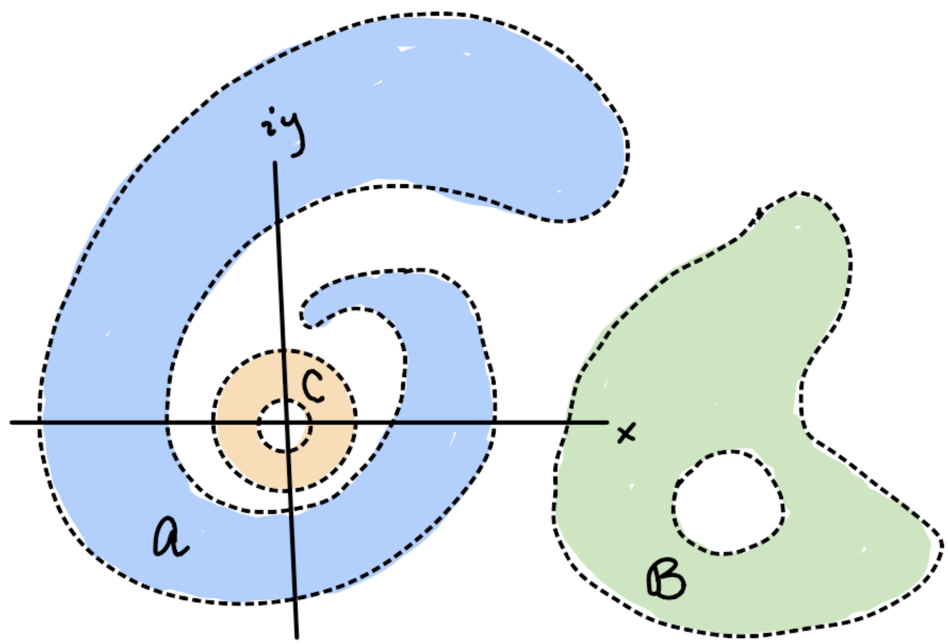
Theorem 2.2.5 (Antiderivative Theorem) If A is *open* and *simply connected*. Let $f : A \rightarrow \mathbb{C}$ be analytic and C^1. Then f has antiderivatives F, unique up to additive constants.

proof: We'll use Cauchy's Theorem to explain heuristically why the path-independence condition (ii) of Theorem 1 holds. Thus antiderivatives exist, and one way to express them is via contour integrals as in the previous discussion:

$$F(z) = \int_{\gamma_{z_0 z}} f(\zeta)\, d\zeta$$

Notice how we will use the "no-holes" idea of *simply-connected*. This explanation is not completely rigorous, but we'll fix that lack of rigor in section 2.3 by defining simply connected more carefully, and also by using different techniques that don't depend on Greens' Theorem and our heuristic pictures of what contours look like.

Example (also relates to alternate way of doing one of the hw exercises due last Friday)
Which of the domains below are *connected?* Which are *simply connected?* Discuss whether it is possible to define $\log(z)$ as an analytic (single-valued) function on each of the domains:



Appendix: Connected domains, path connected domains, simply connected domains: Some Math 3220/Chapter 1.4 analysis background material we need now:

Recall that a domain $A \subseteq \mathbb{C}$ is called *connected* iff there is no disconnection of A into disjoint (relatively) open and non-empty subsets U, V i.e. such that

$$A = U \cup V$$
$$U \cap V = \varnothing.$$

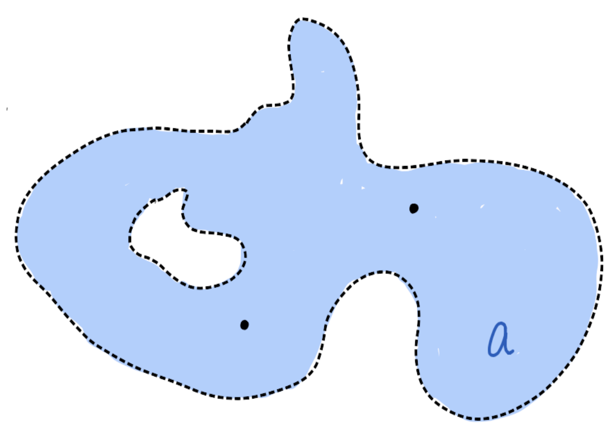
If we restrict to open domains A, then subsets U, V that are relatively open are actually open.

There is a related definition:
Definition A subset $A \subseteq \mathbb{C}$ is called *path connected* iff $\forall\ P, Q \in A$, there exists a continuous path $\gamma : [a, b] \rightarrow A$ such that $\gamma(a) = P,\ \gamma(b) = Q$.

Theorem Let $A \subseteq \mathbb{C}$ be open. Then A is connected if and only if A is path connected. Furthermore, if A is connected then there are piecewise C^1 paths connecting all possible pairs of points in A. (Analogous theorem holds in $\mathbb{R}^n$.)
proof: $\Rightarrow$: Let A be connected and open. We will show it is path connected, with piecewise C^1 paths. Pick any base point $z_0 \in A$. Define U to be the set of points that can be connected to z_0 with a piecewise C^1 path. U is non-empty since $D(z_0; r) \subseteq U$ as long as r is small enough so that the disk is in A. In fact, for all $z \in D(z_0; r)$ we can use the straight-line paths

$$\gamma(t) = z_0 + t(z - z_0), \quad 0 \le t \le 1$$

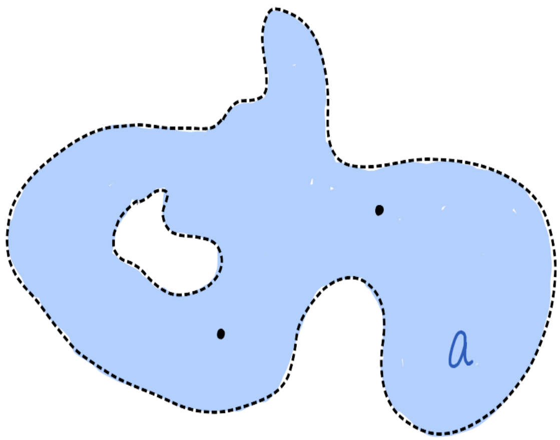
to connect z_0 to z.



The proof that U is open is analogous: Let $z \in U$ and let γ be a piecewise C^1 path connecting z_0 to z. Then for $w \in D(z, r) \subseteq A$ and

$$\gamma_1(t) = z + t(w - z), \quad 0 \le t \le 1$$

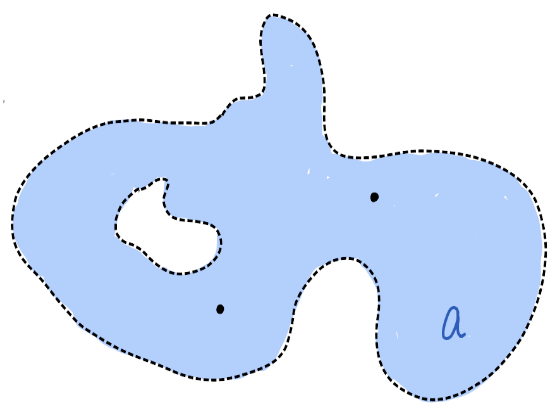
the combined path $\gamma + \gamma_1$ is a piecewise C^1 path connecting z_0 to w. Thus U is open.



But the complement $V := A \setminus U$ is open by a similar argument: If V is non-empty, let $z_1 \in V$, $\mathrm{D}(z_1; r) \subseteq A$. Then $\mathrm{D}(z_1, r) \subseteq V$ as well, since if $\exists\, z \in U \cap D(z_1; r)$ there is a piecewise C^1 path γ from z_0 to z, and letting

$$\gamma_2(t) = z + t(z_1 - z), \quad 0 \le t \le 1,$$

the path $\gamma + \gamma_2$ would connect z_0 to z_1. Thus, since A is connected, we must have that $V = A \setminus U$ is empty.



path connected implies connected:
Let A be path connected. Let $A = U \cup V$ with U, V open, U non-empty, and $U \cap V = \varnothing$. We will show V is empty. If not, pick $P \in U$, $Q \in V$, and let

$$\gamma : [a, b] \to \mathbb{C}$$

be a continuous path connecting P to Q, i.e. $\gamma(a) = P$, $\gamma(b) = Q$. Let $T \in [a, b]$ be defined by

$$T := sup\{t \in [a, b] \mid \gamma([a, t]) \subseteq U\}$$

Because U is open, $T > a$. Because V is open, $T < b$. But if $a < T < b$ then $\gamma(T)$ is in neither U nor V: If $\gamma(T) \in U$ then by continuity and U open, there exists $\delta > 0$ so that $\gamma(\left[T, T + \delta\right) \subseteq U$, hence $\gamma(\left[a, T + \delta\right) \subseteq U$, contradicting the definition of T. Similarly, if $\gamma(T) \in V$, continuity of γ and V open implies there exists $\delta > 0$ so that $\gamma(\left[T - \delta, T\right] \subseteq V$, another contradiction. Thus T can't exist, and V must be empty.